The work of James Maynard

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  - there are infinitely many triples of primes within 433992 of each other.

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- How well can we approximate real numbers by rational ones?
- Theorem (Dirichlet): If $x \in \mathbb{R} \backslash \mathbb{Q}$, then $|x - a/q| < q^{-2}$ for infinitely many pairs $(a, q) \in \mathbb{Z} \times \mathbb{N}$.

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- The conjectured correct exponent is 2.

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- It may be more productive to characterize Diophantine properties probabilistically
- “Metric" Diophantine approximation
- $A := \{x \in [0,1] : |x - a/q| < \psi(q)$ for infinitely many pairs $(a,q) \in \mathbb{Z} \times \mathbb{N}\}$
- Theorem (Khintchine):
  1. If $\sum_q q\psi(q) < \infty$ then $\mathrm{Leb}(A) = 0$.
  2. If $\sum_q q\psi(q) = \infty$ and $q^2\psi(q)$ is decreasing, then $\mathrm{Leb}(A) = 1$.

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- And $A = \{x \in [0,1] : x \in A_q \text{ for infinitely many } q\} = \limsup_{q \to \infty} A_q$.
- Let (X, B, μ) be a probability space, let $A_1, A_2, \ldots$ be measurable sets, and let $A = \limsup_{n \to \infty} A_n$. Then
  1. (The first Borel–Cantelli lemma) If $\sum_{n=1}^{\infty} \mu(A_n) < \infty$ then $\mu(A) = 0$
  2. (The second Borel–Cantelli lemma) If $\sum_{n=1}^{\infty} \mu(A_n) = \infty$ and $A_1, A_2, \ldots$ are pairwise independent, then $\mu(A) = 1$.

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- One instead uses an enhanced version which permits the use of “independence on average"

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- Khintchine's theorem translates to cusp excursions of the geodesic flow on the modular surface
- In this interpretation, the mixing of the geodesic flow provides independence on average

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- No. One can create dependencies using redundancies in denominators
- An explicit example was given by Duffin and Schaeffer in 1941
- Namely, they gave an example of ψ such that $\sum_{q=1}^{\infty} q\psi(q) = \infty$ but $\mu(A) = 0$

The conjecture

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- As before, A^* is the limsup of sets A_q^* which have measure $2\phi(q)\psi(q)$
- Conjecture (Duffin-Schaeffer, 1941) proved by Koukoulopoulos and Maynard in 2020.
  1. If $\sum_q \phi(q)\psi(q) < \infty$ then $\mathrm{Leb}(A^*) = 0$.
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- The proof uses Birkhoff's ergodic theorem applied to multiplication by 2 map on the circle.

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- So S has to be somewhat dense.

- Let q, r be two distinct integers > 2, let $\psi(q), \psi(r) > 0$, and let $M(q,r) = 2max\{\psi(q), \psi(r)\}\,\mathrm{lcm}[q,r]$. If $M(q,r) \leq 1$, then $A_q^* \cap A_r^* = \emptyset$. Otherwise,

$$\mu(A_q^* \cap A_r^*) \ll \phi(q)\psi(q)\phi(r)\psi(r)\exp\left(\sum_{\substack{p|qr/gcd(q,r)\\ p>M(q,r)}} \frac{1}{p}\right).$$

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- Model Problem. Let $D > 1$ and $\delta \in (0,1]$, and let $S \subset [Q, 2Q] \cap \mathbb{Z}$ be a set of $\delta Q/D$ elements such that there are $> \delta \# S^2$ pairs $(q,r) \in S \times S$ with $gcd(q,r) > D$. Must there be an integer $d > D$ that divides $\gg \delta 100 Q/D$ elements of S?

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- An iterative Compression Algorithm inspired by Erdös-Ko-Rado and Dyson.

Thank You!