

*In 2018, Part II of the selection process to the Integrated PhD and PhD programmes at the School of Mathematics, TIFR, Mumbai, consisted of a written examination comprising the following 14 questions.*

1. Let  $G$  be a group. Let  $H$  be the subgroup of  $G$  generated by  $\{x^2 \mid x \in G\}$ . Let  $H'$  be the commutator subgroup of  $G$ , i.e., the subgroup generated by  $\{xyx^{-1}y^{-1} \mid x, y \in G\}$ . Show that  $H \supset H'$ .
2. Let  $p$  be a prime number, and let  $G$  be a finite group whose order is a power of  $p$ . Let  $F$  be a field of characteristic  $p$ , and  $V$  a nonzero vector space over  $F$  equipped with a linear action of  $G$ .<sup>1</sup> Prove that there exists a nonzero subspace  $W \subset V$  such that  $G$  acts trivially on  $W$ .  
**Hint:** One possible strategy involves first proving the result in the special case where  $F$  is the finite field  $\mathbb{F}_p$  and  $V$  is finite dimensional (so that  $V$  is a finite set), and using this special case to prove the general case.
3. Find the characteristic polynomial of  $A^7$ , where  $A$  is the real matrix:

$$\begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix}.$$

4. Let  $S$  be the set of all lines in  $\mathbb{R}^2$  that pass through the origin and have either rational or infinite slope. Show that the usual action of  $\mathrm{SL}_2(\mathbb{Z})$  (the group of  $2 \times 2$  integer matrices with determinant 1) on  $\mathbb{R}^2$  induces a transitive action of  $\mathrm{SL}_2(\mathbb{Z})$  on  $S$ .
5. Consider the ring  $C(0, 1)$  of continuous real-valued functions on  $(0, 1)$ . For any  $P \in (0, 1)$ , write  $\mathfrak{m}_P$  for the ideal:

$$\mathfrak{m}_P := \{f \in C(0, 1) \mid f(P) = 0\}.$$

- (a) Show that  $C(0, 1)$  contains infinitely many maximal ideals that are not of the form  $\mathfrak{m}_P$  for any  $P \in (0, 1)$ .
- (b) Consider the map  $\mathbb{R} \rightarrow C(0, 1)$  which sends  $a \in \mathbb{R}$  to the constant function with value  $a$ . Suppose  $\mathfrak{m} \subset C(0, 1)$  is a maximal ideal with the property that the composition  $\mathbb{R} \rightarrow C(0, 1) \rightarrow C(0, 1)/\mathfrak{m}$  is an isomorphism. Show that  $\mathfrak{m} = \mathfrak{m}_P$  for some  $P \in (0, 1)$ .

6. Suppose  $f$  is a continuous function on  $\mathbb{R}$  such that we have:

$$\forall x \in \mathbb{R}, \quad f(x) = f(x^2).$$

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<sup>1</sup>In other words, we have an action  $G \times V \rightarrow V$  of the group  $G$  on the set  $V$ , written  $(g, v) \mapsto g \cdot v$ , such that for all  $g \in G$ , the map  $v \mapsto g \cdot v$  is an  $F$ -linear transformation of the  $F$ -vector space  $V$ .

Show that  $f$  is constant.

7. Let  $f$  be a continuous real-valued function on  $[0, 1]$ . Show that:

$$\lim_{n \rightarrow \infty} \int_0^1 f(x) \sin(nx) dx = 0.$$

8. Suppose  $f : [0, 1] \rightarrow [0, 1]$  is a nondecreasing function, which may or may not be continuous. Show that there exists  $x \in [0, 1]$  such that  $f(x) = x$ .

9. Let  $D : \mathbb{C}[x] \rightarrow \mathbb{C}[x]$  be the  $\mathbb{C}$ -linear map given by  $Df = df/dx$ . Let  $e^D : \mathbb{C}[x] \rightarrow \mathbb{C}[x]$  be the induced linear map given by:

$$f \mapsto (e^D)(f) = \sum_{n \geq 0} \frac{1}{n!} D^n(f).$$

(a) Show that the above formula does give a well-defined map, which is linear.

(b) Compute  $\ker(e^D)$  and  $\text{coker}(e^D)$ .

10. Prove or disprove: any bijective continuous function  $(0, 1) \rightarrow \mathbb{R}$  has a continuous inverse.

11. Prove or disprove:  $\mathbb{R}^3 \setminus \{(0, 0, 0)\}$  and  $\mathbb{R}^3 \setminus B_\varepsilon$  are homeomorphic, where  $\varepsilon > 0$  is a real number and  $B_\varepsilon$  is the closed ball given by  $\{(x, y, z) \in \mathbb{R}^3 \mid \sqrt{x^2 + y^2 + z^2} \leq \varepsilon\}$ .

12. Prove or disprove the following assertion. Let  $S$  be a subset of  $\mathbb{R}$ , with the induced order, such that every nonempty subset of  $S$  contains a minimum element (in other words,  $S$  is well-ordered). Then  $S$  is necessarily countable (i.e., empty, finite or countably infinite).

13. Given any two ellipses in  $\mathbb{R}^2$ , show that there exists a linear (but not necessarily orthogonal) change of coordinates ( $x' = ax + by + k$ ,  $y' = cx + dy + l$ ;  $ad - bc \neq 0$ ) so that both the ellipses have axes parallel to the new coordinate axes - namely, under the new coordinate system, each of them can be described by an equation of the form:

$$\frac{(x' - \alpha)^2}{r^2} + \frac{(y' - \beta)^2}{s^2} = 1.$$

14. For any  $n \geq 1$ , let  $P_n(x) = \frac{d^n}{dx^n} (x^n(1-x)^n)$ . Show that for  $m \neq n$ :

$$\int_0^1 P_n(x) P_m(x) dx = 0.$$