Questions Asked in GS 2019 (Mathematics), Part-II

- There are 11 problems in this paper. Problem 1 is worth 20 points. All other problems are worth 10 points.
- Some of the problems have multiple parts. To solve one part of the problem, the students were allowed to use any of the previous parts, even if they were unable to solve the previous parts.

1. (a) (5 points.) Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a continuous function such that the inverse image of a bounded set is bounded. Prove that for every closed subset $Z \subseteq \mathbb{R}^m$, $f(Z)$ is closed in $\mathbb{R}^n$.

   (b) (3 points.) If $z \in \mathbb{C}$ is a root of the monic polynomial $t^n + a_1 t^{n-1} + \cdots + a_n \in \mathbb{C}[t]$, show that:

   $$|z| \leq \max\{1, |a_1| + \cdots + |a_n|\}.$$

   (c) (3 points.) Let $\phi : \mathbb{C}^3 \to \mathbb{C}^3$ be the map $(x, y, z) \mapsto (x + y + z, xy + yz + zx, xyz)$. Show that ϕ is surjective.

   (d) (4 points.) Prove that a subset $Z \subseteq \mathbb{C}^3$ is closed if and only if $\phi^{-1}(Z)$ is closed, where ϕ is as defined in part (c).

   (e) (5 points.) Prove that the map ϕ defined in part (c) is both a closed and an open map, i.e. the image under ϕ of closed sets is closed and the image of open sets is open.

2. Let R be an integral domain. A subring A of the polynomial ring $R[x_1, \cdots, x_n]$ contains two polynomials f, g of mutually coprime degrees. Prove that for every large enough integer N the subring A contains a polynomial of degree N.

3. Let $z_1, z_2, \cdots z_n \in \mathbb{C}$ be such that the real and imaginary parts of each z_i are non-negative. Show that

   $$\left| \sum_{i=1}^{n} z_i \right| \geq \frac{1}{\sqrt{2}} \sum_{i=1}^{n} |z_i|.$$

4. Does there exist a positive integer n such that the decimal representation of 3^n starts with the digits 2019? Justify your assertion.

5. Does -1 have a square-root in the ring $\mathbb{R}[x]/(x^2+1)^2$? Justify your assertion.

6. Prove that the system of n linear equations in m variables

$$a_{i1}x_1 + \cdots + a_{im}x_m = b_i \text{ for all } i \in \{1, \cdots, n\},$$

where each $a_{ij} \in \mathbb{Q}$ and each $b_i \in \mathbb{Q}$, has a solution in $\mathbb{C}^m$ if and only if it has a solution in $\mathbb{Q}^m$.

7. Let $v : \mathbb{R} \to \mathbb{R}^2$ be a differentiable function such that the velocity vector $\frac{dv}{dt} \neq 0$ at all $t \in \mathbb{R}$. Prove that v is not surjective.

8. Let V be a vector space over $\mathbb{Q}$ of countably infinite dimension. Let V^* be its dual vector space of $\mathbb{Q}$-linear functionals $\lambda : V \to \mathbb{Q}$. Are V and V^* isomorphic as $\mathbb{Q}$-vector spaces? Justify your assertion.

9. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function. Prove that for any $a, b \in \mathbb{R}$

$$\left(\int_a^b \sqrt{1 + f'(x)^2} dx \right)^2 \geq (b-a)^2 + (f(b) - f(a))^2.$$

10. (a) (3 points.) Let G be any group and let $H \leq G$ be a finite index subgroup. Then prove that there exists a finite index normal subgroup $N \leq G$ such that $N \leq H$.
    (b) (7 points.) Let G be a finitely generated group. Prove that for each positive integer k there are only finitely many subgroups $H \leq G$ of index k.

11. Does there exist a sequence of continuous functions $f_n : [0,1] \to [0, \infty)$ such that

$$\lim_{n \to \infty} \int_0^1 f_n(x) dx = 0$$

but there does not exist any $x \in [0,1]$ for which the sequence $\{f_n(x)\}$ converges? Justify your assertion.