Atiyah-Singer Revisited

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Five lectures:

1. Dirac operator✓
2. Atiyah-Singer revisited
3. What is K-homology?
4. Beyond ellipticity
5. The Riemann-Roch theorem

ATIYAH-SINGER REVISITED

This is an expository talk about the Atiyah-Singer index theorem.

1. Dirac operator of $\mathbb{R}^n$ will be defined.✓
2. Some low dimensional examples of the theorem will be considered.
3. A special case of the theorem will be proved, with the proof based on Bott periodicity.
4. The proof will be outlined that the special case implies the full theorem.

Atiyah-Singer Index theorem

M compact C^∞ manifold without boundary

D an elliptic differential (or elliptic pseudo-differential) operator on M

$E^0, E^1, \quad C^\infty \quad \mathbb{C}$ vector bundles on M

$C^\infty(M, E^j)$ denotes the $\mathbb{C}$ vector space of all C^∞ sections of E^j.

$D\colon C^\infty(M, E^0) \longrightarrow C^\infty(M, E^1)$

D is a linear transformation of $\mathbb{C}$ vector spaces.

Atiyah-Singer Index theorem

M compact C^∞ manifold without boundary

D an elliptic differential (or elliptic pseudo-differential) operator on M

Index$(D) := \dim_{\mathbb{C}}$ (Kernel D) $- \dim_{\mathbb{C}}$ (Cokernel D)

Theorem (M.Atiyah and I.Singer)

Index (D) = (a topological formula)

Example

$M = S^1 = \{(t_1, t_2) \in \mathbb{R}^2 \mid t_1^2 + t_2^2 = 1\}$

$D_f \colon L^2(S^1) \longrightarrow L^2(S^1)$ is

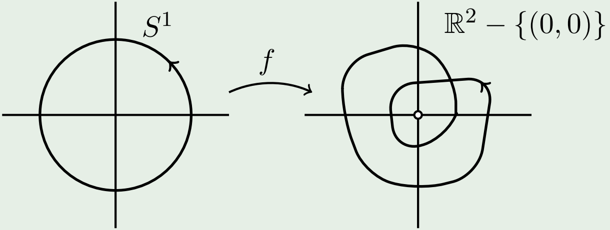
T_f	0
0	I

where $L^2(S^1) = L^2_+(S^1) \oplus L^2_-(S^1)$.

$L^2_+(S^1)$ has as orthonormal basis $e^{in\theta}$ with $n = 0, 1, 2, \ldots$
$L^2_-(S^1)$ has as orthonormal basis $e^{in\theta}$ with $n = -1, -2, -3, \ldots$.

Example

$f\colon S^1 \longrightarrow \mathbb{R}^2 - \{0\}$ is a C^∞ map.



$T_f\colon L^2_+(S^1) \longrightarrow L^2_+(S^1)$ is the composition
$L^2_+(S^1) \xrightarrow{\mathcal{M}_f} L^2(S^1) \longrightarrow L^2_+(S^1)$

$T_f\colon L^2_+(S^1) \longrightarrow L^2_+(S^1)$ is the Toeplitz operator associated to f

Example

Thus T_f is the composition

$$T_f \colon L^2_+(S^1) \xrightarrow{\mathcal{M}_f} L^2(S^1) \xrightarrow{P} L^2_+(S^1)$$

where $L^2_+(S^1) \xrightarrow{\mathcal{M}_f} L^2(S^1)$ is $v \mapsto fv$

$$fv(t_1, t_2) := f(t_1, t_2)v(t_1, t_2) \qquad \forall (t_1, t_2) \in S^1 \qquad \mathbb{R}^2 = \mathbb{C}$$

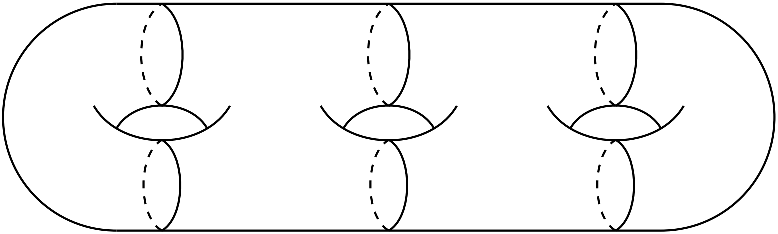
and $L^2(S^1) \xrightarrow{P} L^2_+(S^1)$ is the Hilbert space projection.

$$D_f(v + w) := T_f(v) + w \qquad v \in L^2_+(S^1), \quad w \in L^2_-(S^1)$$

Index(D_f) = -winding number (f).

RIEMANN - ROCH

M compact connected Riemann surface



$$\text{genus of } M = \ \# \text{ of holes}$$
$$= \frac{1}{2}\,[\mathrm{rank} H_1(M;\mathbb{Z})]$$

D a divisor of M

D consists of a finite set of points of M $p_1, p_2, \ldots, p_l$ and an integer assigned to each point $n_1, n_2, \ldots, n_l$

Equivalently

D is a function $D\colon M \to \mathbb{Z}$ with finite support

Support$(D) = \{p \in M \mid D(p) \neq 0\}$

Support(D) is a finite subset of M

D a divisor on M

$\deg(D) := \sum_{p \in M} D(p)$

Remark

D_1, D_2 two divisors

$$D_1 \geqq D_2 \text{ iff } \forall p \in M, D_1(p) \geqq D_2(p)$$

Remark

D a divisor, $-D$ is

$$(-D)(p) = -D(p)$$

Example

Let $f\colon M \to \mathbb{C} \cup \{\infty\}$ be a meromorphic function.

Define a divisor $\delta(f)$ by:

$$\delta(f)(p) = \begin{cases} 0 \text{ if } p \text{ is neither a zero nor a pole of } f \\ \text{order of the zero if } f(p) = 0 \\ -\text{(order of the pole) if } p \text{ is a pole of } f \end{cases}$$

Example

Let w be a meromorphic 1-form on M. Locally w is $f(z)dz$ where f is a (locally defined) meromorphic function. Define a divisor $\delta(w)$ by:

$$\delta(w)(p) = \begin{cases} 0 \text{ if } p \text{ is neither a zero nor a pole of } w \\ \text{order of the zero if } w(p) = 0 \\ -(\text{order of the pole}) \text{ if } p \text{ is a pole of } w \end{cases}$$

D a divisor on M

$$H^0(M,D) := \left\{ \begin{array}{l} \text{meromorphic functions} \\ f\colon M \to \mathbb{C} \cup \{\infty\} \end{array} \middle| \, \delta(f) \geqq -D \right\}$$

$$H^1(M,D) := \left\{ \begin{array}{l} \quad \text{meromorphic 1-forms} \\ w \text{ on } M \end{array} \middle| \, \delta(w) \geqq D \right\}$$

Lemma

$H^0(M,D)$ and $H^1(M,D)$ are finite dimensional $\mathbb{C}$ vector spaces

$$\dim_{\mathbb{C}} H^0(M,D) < \infty$$
$$\dim_{\mathbb{C}} H^1(M,D) < \infty$$

Theorem (R. R.)

Let M be a compact connected Riemann surface and let D be a divisor on M. Then:

$$\dim_{\mathbb{C}} H^0(M, D) - \dim_{\mathbb{C}} H^1(M, D) = d - g + 1$$

$$d = \textit{degree}\ (D)$$
$$g = \textit{genus}\ (M)$$

HIRZEBRUCH-RIEMANN-ROCH

M non-singular projective algebraic variety / $\mathbb{C}$

E an algebraic vector bundle on M

$\underline{E}$ = sheaf of germs of algebraic sections of E

$H^j(M, \underline{E}) :=$ j-th cohomology of M using $\underline{E}$,
$j = 0, 1, 2, 3, \ldots$

Equivalently, $H^j(M, \underline{E})$ is the j-th homology of the Dolbeault complex of E.

LEMMA

For all $j = 0, 1, 2, \ldots \dim_{\mathbb{C}} H^j(M, \underline{E}) < \infty$.

For all $j > \dim_{\mathbb{C}}(M), \quad H^j(M, \underline{E}) = 0$.

$$\chi(M, E) := \sum_{j=0}^{n} (-1)^j \dim_{\mathbb{C}} H^j(M, \underline{E})$$

$n = \dim_{\mathbb{C}}(M)$

THEOREM[HRR] Let M be a non-singular projective algebraic variety / $\mathbb{C}$ and let E be an algebraic vector bundle on M. Then

$$\chi(M, E) = (ch(E) \cup Td(M))[M]$$

Hirzebruch-Riemann-Roch

Theorem (HRR)

Let M *be a non-singular projective algebraic variety* $/\,\mathbb{C}$ *and let* E *be an algebraic vector bundle on* M. *Then*

$$\chi(M, E) = (ch(E) \cup Td(M))[M]$$

$1, \varepsilon, \varepsilon^2, \varepsilon^3, \varepsilon^4$
5 Eckp. des reg 5-Ecks
A_5
$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \begin{array}{l} ad - bc = 1, \\ a, b, c, d \in F_5 \end{array} \right\}$

SPECIAL CASE OF ATIYAH-SINGER

Let M be a compact even-dimensional Spin^c manifold without boundary. Let E be a $\mathbb{C}$ vector bundle on M.

D_E denotes the Dirac operator of M tensored with E.

$$D_E\colon C^\infty(M, S^+ \otimes E) \longrightarrow C^\infty(M, S^- \otimes E)$$

S^+, S^- are the positive (negative) spinor bundles on M.

THEOREM Index$(D_E) = (ch(E) \cup Td(M))[M]$.

$K_0(\cdot)$

Definition

Define an abelian group denoted $K_0(\cdot)$ by considering pairs (M, E) such that:

1. M is a compact even-dimensional Spin^c manifold without boundary.
2. E is a $\mathbb{C}$ vector bundle on M.

Set $\mathrm{K}_0(\cdot) = \{(M, E)\}/\sim$ where the the equivalence relation $\sim$ is generated by the three elementary steps

- Bordism
- Direct sum - disjoint union
- Vector bundle modification

Addition in $\mathrm{K}_0(\cdot)$ is disjoint union.

$$(M, E) + (M', E') = (M \sqcup M', E \sqcup E')$$

In $\mathrm{K}_0(\cdot)$ the additive inverse of (M, E) is $(-M, E)$ where $-M$ denotes M with the Spin^c structure reversed.

$$-(M, E) = (-M, E)$$

Isomorphism (M, E) is isomorphic to (M', E') iff $\exists$ a diffeomorphism

$$\psi\colon M \to M'$$

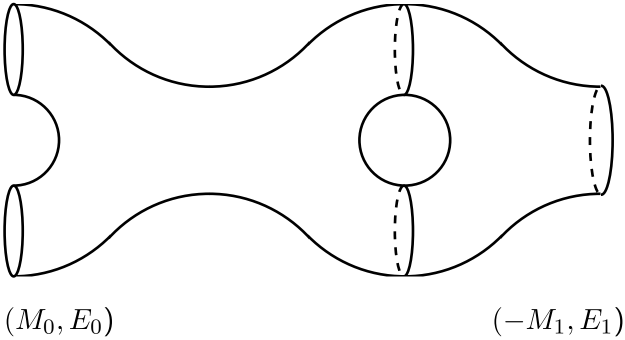
preserving the Spin^c-structures on M, M' and with

$$\psi^*(E') \cong E.$$

Bordism (M_0, E_0) is **bordant** to (M_1, E_1) iff $\exists$ (Ω, E) such that:

1. Ω is a compact odd-dimensional Spin^c manifold with boundary.
2. E is a $\mathbb{C}$ vector bundle on Ω.
3. $(\partial\Omega, E|_{\partial\Omega}) \cong (M_0, E_0) \sqcup (-M_1, E_1,)$

$-M_1$ is M_1 with the Spin^c structure reversed.



(M_0, E_0)
$(-M_1, E_1)$

Direct sum - disjoint union

Let E, E' be two $\mathbb{C}$ vector bundles on M

$$(M, E) \sqcup (M, E') \sim (M, E \oplus E')$$

Vector bundle modification

(M, E)

Let F be a Spin^c vector bundle on M

Assume that

$$\dim_{\mathbb{R}}(F_p) \equiv 0 \mod 2 \quad p \in M$$

for every fiber F_p of F

$$\mathbf{1}_{\mathbb{R}} = M \times \mathbb{R}$$

$$S(F \oplus \mathbf{1}_{\mathbb{R}}) := \text{unit sphere bundle of } F \oplus \mathbf{1}_{\mathbb{R}}$$

$$(M, E) \sim (S(F \oplus \mathbf{1}_{\mathbb{R}}), \beta \otimes \pi^* E)$$

$$\begin{array}{c} S(F \oplus \mathbf{1}_{\mathbb{R}}) \\ \Big\downarrow \pi \\ M \end{array}$$

This is a fibration with even-dimensional spheres as fibers.

$F \oplus \mathbf{1}_{\mathbb{R}}$ is a Spin^c vector bundle on M with odd-dimensional fibers.

The Spin^c structure for F causes there to appear on $S(F \oplus 1_{\mathbb{R}})$ a $\mathbb{C}$-vector bundle β whose restriction to each fiber of π is the Bott generator vector bundle of that oriented even-dimensional sphere.

$$(M, E) \sim (S(F \oplus \mathbf{1}_{\mathbb{R}}), \beta \otimes \pi^* E)$$

Addition in $\mathrm{K}_0(\cdot)$ is disjoint union.

$$(M, E) + (M', E') = (M \sqcup M', E \sqcup E')$$

In $\mathrm{K}_0(\cdot)$ the additive inverse of (M, E) is $(-M, E)$ where $-M$ denotes M with the Spin^c structure reversed.

$$-(M, E) = (-M, E)$$

DEFINITION. (M, E) *bounds* $\iff \exists\, (\Omega, \widetilde{E})$ with :

1. Ω is a compact odd-dimensional Spin^c manifold with boundary.
2. $\widetilde{E}$ is a $\mathbb{C}$ vector bundle on Ω.
3. $(\partial\Omega, \widetilde{E}|_{\partial\Omega}) \cong (M, E)$

REMARK. $(M, E) = 0$ in $K_0(\cdot) \iff (M, E) \sim (M', E')$ where (M', E') bounds.

Consider the homomorphism of abelian groups

$$\begin{aligned} K_0(\cdot) &\longrightarrow \mathbb{Z} \\ (M, E) &\longmapsto \mathrm{Index}(D_E) \end{aligned}$$

Notation

D_E is the Dirac operator of M tensored with E.

$$\begin{aligned} K_0(\cdot) &\longrightarrow \mathbb{Z} \\ (M, E) &\longmapsto \operatorname{Index}(D_E) \end{aligned}$$

It is a corollary of Bott periodicity that this homomorphism of abelian groups is an isomorphism.

Equivalently, $\text{Index}(D_E)$ is a complete invariant for the equivalence relation generated by the three elementary steps; i.e. $(M, E) \sim (M', E')$ if and only if $\mathrm{Index}(D_E) = \mathrm{Index}(D'_{E'})$.

Have three problems for :

$$\begin{aligned} K_0(\cdot) &\longrightarrow \mathbb{Z} \\ (M, E) &\longmapsto \mathrm{Index}(D_E) \end{aligned}$$

(i) well-defined

(ii) surjective

(iii) injective

In order to prove that the homomorphism of abelian groups

$$\begin{aligned} \mathrm{K}_0(\cdot) &\longrightarrow \mathbb{Z} \\ (M, E) &\longmapsto \mathrm{Index}(D_E) \end{aligned}$$

is well-defined, the three elementary moves (bordism, direct sum - disjoint union, vector bundle modification) must be proved to be index-preserving.

Proof that $K_0(\cdot) \longrightarrow \mathbb{Z}$ is well-defined.

For bordism-invariance of the index have three proofs :

1. Proof in R. Palais book *Seminar on the Atiyah-Singer Index Theorem* (1965).
2. Proof by M. S. Raghunathan in paper "The Atiyah-Singer Index Theorem", Contemporary Mathematics (2008). Uses Morse theory to decompose any given bordism into elementary bordisms. Uses existence of the index density.
3. Proof using Atiyah-Kasparov K-homology. Atiyah-Kasparov K-homology will be defined in the next lecture.

Proof that $K_0(\cdot) \longrightarrow \mathbb{Z}$ is well-defined.

For vector bundle modification :
Start with (M, E). Given a Spin^c vector bundle F on M with fiber dimension$_{\mathbb{R}}(F)$ even, — form $\left(S(F \oplus \mathbf{1}_{\mathbb{R}}), \beta \otimes \pi^* E\right)$

$$\begin{array}{c} S(F \oplus \mathbf{1}_{\mathbb{R}}) \\ \big\downarrow \pi \\ M \end{array}$$

π is a fibration with oriented even-dimensional spheres as fibers. Restriction of β to each fiber of π is the Bott generator vector bundle of that oriented even-dimensional sphere.

$$\begin{array}{c} S(F \oplus \mathbf{1}_{\mathbb{R}}) \\ \Big\downarrow \pi \\ M \end{array}$$

For each fiber of π can form the elliptic operator (Dirac of the fiber)$\otimes$(β restricted to the fiber).
Thus for each point $p \in M$ have an elliptic operator.
Hence have a family of elliptic operators over M .
Key point is that the index of this family is the trivial line bundle on M — i.e. is $M \times \mathbb{C}$.

This proof that vector bundle modification is index-preserving is in essence the same as :

- Proof by Atiyah and Singer of compatibility of index and Thom isomorphism.
- Proof by M. S. Raghunathan that reduces Atiyah-Singer to the special case when the manifold is stably parallelizable.

BOTT PERIODICITY

$$\pi_j GL(n, \mathbb{C}) = \begin{cases} \mathbb{Z} & \text{j odd} \\ \\ 0 & \text{j even} \end{cases}$$

$$j = 0, 1, 2, \ldots, 2n - 1$$

Why does Bott periodicity imply that

$$\begin{aligned} \mathrm{K}_0(\cdot) &\longrightarrow \mathbb{Z} \\ (M, E) &\longmapsto \mathrm{Index}(D_E) \end{aligned}$$

is an isomorphism?

To prove surjectivity must find an (M, E) with $\mathrm{Index}(D_E) = 1$.

e.g. Let $M = \mathbb{C}P^n$, and let E
be the trivial (complex) line bundle on $\mathbb{C}P^n$
$E{=}1_{\mathbb{C}} = \mathbb{C}P^n \times \mathbb{C}$
$\mathrm{Index}(\mathbb{C}P^n, 1_{\mathbb{C}}) = 1$

Thus Bott periodicity is not used in the proof of surjectivity.

Lemma used in the Proof of Injectivity

Given any (M, E) there exists an even-dimensional sphere S^{2n} and a $\mathbb{C}$-vector bundle F on S^{2n} with $(M, E) \sim (S^{2n}, F)$.

Bott periodicity is not used in the proof of this lemma.
The lemma is proved by a direct argument using the definition of the equivalence relation on the pairs (M, E).

Let r be a positive integer, and let $\mathrm{Vect}_{\mathbb{C}}(S^{2n}, r)$
be the set of isomorphism classes of $\mathbb{C}$ vector bundles on S^{2n}
of rank r, i.e. of fiber dimension r.

$$\mathrm{Vect}_{\mathbb{C}}(S^{2n}, r) \longleftrightarrow \pi_{2n-1} GL(r, \mathbb{C})$$

PROOF OF INJECTIVITY

Let (M, E) have $\operatorname{Index}(M, E) = 0$.
By the above lemma, we may assume that $(M, E) = (S^{2n}, F)$.
Using Bott periodicity plus the bijection

$$\operatorname{Vect}_{\mathbb{C}}(S^{2n}, r) \longleftrightarrow \pi_{2n-1}GL(r, \mathbb{C})$$

we may assume that F is of the form

$$F = \theta^p \oplus q\beta$$

$\theta^p = S^{2n} \times \mathbb{C}^p$ and β is the Bott generator vector bundle on S^{2n}.
Convention. If $q < 0$, then $q\beta = |q|\beta^*$.

$\mathrm{Index}(S^{2n}, \beta) = 1 \quad \mathrm{Index}(S^{2n}, \theta^p) = 0$
Therefore

$$\mathrm{Index}(S^{2n}, F) = 0 \Longrightarrow q = 0$$

Hence $(S^{2n}, F) = (S^{2n}, \theta^p)$. This bounds

$$(S^{2n}, \theta^p) = \partial(B^{2n+1}, B^{2n+1} \times \mathbb{C}^p)$$

and so is zero in $\mathrm{K}_0(\cdot)$.
QED

Define a homomorphism of abelian groups

$$\begin{aligned} \mathrm{K}_0(\cdot) &\longrightarrow \mathbb{Q} \\ (M, E) &\longmapsto \big(ch(E) \cup \mathrm{Td}(M)\big)[M] \end{aligned}$$

where $ch(E)$ is the Chern character of E and $\mathrm{Td}(M)$ is the Todd class of M.

$ch(E) \in H^*(M, \mathbb{Q})$ and $\mathrm{Td}(M) \in H^*(M, \mathbb{Q})$.

$[M]$ is the orientation cycle of M. $[M] \in H_*(M, \mathbb{Z})$.

Granted that

$$\begin{aligned} \mathrm{K}_0(\cdot) &\longrightarrow \mathbb{Z} \\ (M,E) &\longmapsto \mathrm{Index}(D_E) \end{aligned}$$

is an isomorphism, to prove that these two homomorphisms are equal, it suffices to check one example (M,E) with Index$(D_E) = 1$.

Reference. P. F. Baum and E. van Erp, *K-homology and Fredholm Operators I : Dirac Operators*, to appear.

Symbol of a differential operator

Let Y be a C^∞ manifold (possibly with boundary).
Y is not required to be oriented.
Y is not required to be even dimensional.
On Y let

$$\delta : C^\infty(Y, E^0) \longrightarrow C^\infty(Y, E^1)$$

be a differential operator of order k.
Denote by $\pi \colon T^*Y \to Y$ the projection $T^*Y \to Y$.
The symbol (or principal symbol) of δ is for each $\xi \in T^*Y$ a $\mathbb{C}$-linear map

$$\sigma(\xi) \colon E^0_{\pi(\xi)} \longrightarrow E^1_{\pi(\xi)}$$

defined as follows :

Symbol of a differential operator

$$\delta : C^\infty(Y, E^0) \longrightarrow C^\infty(Y, E^1) \qquad\qquad k = \mathrm{order}\ (\delta)$$

Given $\xi \in T^*Y$ and $u \in E^0_{\pi(\xi)}$, set $p = \pi(\xi)$, and choose :
(i) $s \in C^\infty(Y, E^0)$ with $s(p) = u$.
(ii) a C^∞ function $f\colon Y \to \mathbb{R}$ with $f(p) = 0$ and $df(p) = \xi$.

Then:

$$\sigma(\xi)(u) := (\frac{1}{k!})\delta(f^k s)(p)$$

$\sigma(\xi)\colon E^0_p \to E^1_p$ does not depend on the choices (i) (ii).

Symbol of a differential operator

$$\delta : C^\infty(Y, E^0) \longrightarrow C^\infty(Y, E^1)$$

The differential operator δ is elliptic if for every non-zero $\xi \in T^*Y$

$$\sigma(\xi)\colon E^0_{\pi(\xi)} \to E^1_{\pi(\xi)}$$

is an isomorphism.
The symbol σ of δ can be viewed as a vector bundle map

$$\sigma\colon \pi^* E^0 \to \pi^* E^1$$

This basic theory (i.e. symbol, elliptic etc.) extends to pseudo-differential operators.

Let X be a compact C^∞ manifold without boundary.
X is not required to be oriented.
X is not required to be even dimensional.
On X let

$$\delta : C^\infty(X, E^0) \longrightarrow C^\infty(X, E^1)$$

be an elliptic differential (or elliptic pseudo-differential) operator.

$(S(T^*X \oplus 1_{\mathbb{R}}), E_\sigma) \in \mathrm{K}_0(\cdot)$, and

$$\mathrm{Index}(D_{E_\sigma}) = \mathrm{Index}(\delta).$$

$$(S(T^*X \oplus 1_{\mathbb{R}}), E_\sigma)$$

$$\Downarrow$$

$$\text{Index}(\delta) = (ch(E_\sigma) \cup \mathrm{Td}((S(T^*X \oplus 1_{\mathbb{R}})))[(S(T^*X \oplus 1_{\mathbb{R}})]$$

and this is the general Atiyah-Singer formula.

$S(T^*X \oplus 1_{\mathbb{R}})$ is the unit sphere bundle of $T^*X \oplus 1_{\mathbb{R}}$.
$S(T^*X \oplus 1_{\mathbb{R}})$ is even dimensional and is — in a natural way — a Spin^c manifold.

E_σ is the $\mathbb{C}$ vector bundle on $S(T^*X \oplus 1_{\mathbb{R}})$ obtained by doing a clutching construction using the symbol σ of δ.

Construction of E_σ

upper hemisphere lower hemisphere

$$S(T^*X \oplus 1_{\mathbb{R}}) = B_+(T^*X \oplus 1_{\mathbb{R}}) \cup_{S(T^*X)} B_-(T^*X \oplus 1_{\mathbb{R}})$$

$$E_\sigma := \pi^*(E^0) \cup_\sigma \pi^*(E^1)$$

$$(S(T^*X \oplus 1_{\mathbb{R}}), E_\sigma) \in \mathrm{K}_0(\cdot)$$

$$\mathrm{Index}(D_{E_\sigma}) = \mathrm{Index}(\delta)$$

Proof. Show that can go from δ to D_{E_σ} by an explicit finite sequence of index-preserving moves. This uses pseudo-differential operators.

Reference. P. F. Baum and E. van Erp, *K-homology and Fredholm Operators II : Elliptic Operators*, to appear.

Next lecture : Tomorrow (i.e. Wednesday, 5 August).