WHAT IS K-HOMOLOGY ?

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Five lectures:

1. Dirac operator✓
2. Atiyah-Singer revisited✓
3. What is K-homology?
4. Beyond ellipticity
5. The Riemann-Roch theorem

Define a homomorphism of abelian groups

$$\begin{aligned} \mathrm{K}_0(\cdot) &\longrightarrow \mathbb{Q} \\ (M, E) &\longmapsto \big(ch(E) \cup \mathrm{Td}(M)\big)[M] \end{aligned}$$

where $ch(E)$ is the Chern character of E and $\mathsf{Td}(M)$ is the Todd class of M.

$ch(E) \in H^*(M, \mathbb{Q})$ and $\mathsf{Td}(M) \in H^*(M, \mathbb{Q})$.

$[M]$ is the orientation cycle of M. $[M] \in H_*(M, \mathbb{Z})$.

Bott periodicity implies that

$$\begin{aligned} \mathrm{K}_0(\cdot) &\longrightarrow \mathbb{Z} \\ (M, E) &\longmapsto \mathrm{Index}(D_E) \end{aligned}$$

is an isomorphism. Hence to prove that these two homomorphisms are equal, it suffices to check one example with Index$(M, E) = 1$.

Let X be a compact C^∞ manifold without boundary.
X is not required to be oriented.
X is not required to be even dimensional.
On X let

$$\delta : C^\infty(X, E^0) \longrightarrow C^\infty(X, E^1)$$

be an elliptic differential (or pseudo-differential) operator. Then:

$$(S(T^*X \oplus 1_{\mathbb{R}}), E_\sigma) \in \mathrm{K}_0(\cdot)$$

and

$$\mathrm{Index}(D_{E_\sigma}) = \mathrm{Index}(\delta).$$

$$(S(T^*X \oplus 1_{\mathbb{R}}), E_\sigma) \in \mathrm{K}_0(\cdot) \text{ and } \mathrm{Index}(D_{E_\sigma}) = \mathrm{Index}(\delta)$$

$$\Downarrow$$

$$\mathsf{Index}(\delta) = (ch(E_\sigma) \cup \mathrm{Td}(S(T^*X \oplus 1_{\mathbb{R}})))[S(T^*X \oplus 1_{\mathbb{R}})]$$

and this is the general Atiyah-Singer formula.

$S(T^*X \oplus 1_{\mathbb{R}})$ is the unit sphere bundle of $T^*X \oplus 1_{\mathbb{R}}$.
$S(T^*X \oplus 1_{\mathbb{R}})$ is even dimensional and is — in a natural way — a Spin^c manifold.

E_σ is the $\mathbb{C}$ vector bundle on $S(T^*X \oplus 1_{\mathbb{R}})$ obtained by doing a clutching construction using the (principal) symbol σ of δ.

FACT:
If M is any closed odd-dimensional C^∞ manifold and δ is any elliptic differential operator on M, then

$$\mathrm{Index}(\delta) = 0$$

QUESTION :
What are the examples on odd-dimensional closed C^∞ manifolds for the Atiyah-Singer index theorem?

"closed" $=$ "compact without boundary"

Let M be a closed odd-dimensional Spin^c manifold.
The Dirac operator D of M

$$D\colon C^\infty(M,\mathcal{S}) \longrightarrow C^\infty(M,\mathcal{S})$$

is a symmetric operator which has a unique self-adjoint extension, and thus can be viewed as a self-adjoint unbounded operator on the Hilbert space $L^2(M,\mathcal{S})$.

The spectrum of D consists of real numbers $\lambda_1, \lambda_2, \lambda_3, \ldots$
Each λ_j is an isolated point in the spectrum.
Each λ_j is an eigenvalue whose eigenspace $E(\lambda_j)$ is finite-dimensional and is contained in $C^\infty(M,\mathcal{S})$.

$$E(\lambda_j) \subset C^\infty(M,\mathcal{S})$$

$$\dim_{\mathbb{C}} E(\lambda_j) < \infty$$

Let $L^2_+(M,\mathcal{S})$ $\quad(L^2_-(M,\mathcal{S}))$ be the sub Hilbert space of $L^2(M,\mathcal{S})$ spanned by the $E(\lambda_j)$ with $\lambda_j \geq 0$ $\quad(\lambda_j < 0)$.

The Hilbert space $L^2(M,\mathcal{S})$ is then the direct sum :

$$L^2(M,\mathcal{S}) = L^2_+(M,\mathcal{S}) \oplus L^2_-(M,\mathcal{S})$$

Let $f\colon M \to \mathbb{C}$ be a C^∞ function. The multiplication operator

$$\mathcal{M}_f\colon L^2(M,\mathcal{S}) \to L^2(M,\mathcal{S})$$

is

$$\mathcal{M}_f(s) = fs \qquad (fs)(p) = f(p)s(p) \qquad s \in L^2(M,\mathcal{S}) \quad p \in M$$

$f\colon M \to \mathbb{C}$
The Toeplitz operator T_f associated to f

$$T_f\colon L^2_+(M, \mathcal{S}) \to L^2_+(M, \mathcal{S})$$

is the compression of $\mathcal{M}_f$ to $L^2_+(M, \mathcal{S})$ i.e. T_f is the composition

$$T_f\colon L^2_+(M, \mathcal{S}) \xrightarrow{\mathcal{M}_f} L^2(M, \mathcal{S}) \xrightarrow{P} L^2_+(M, \mathcal{S})$$

where $P\colon L^2(M, \mathcal{S}) \to L^2_+(M, \mathcal{S})$ is the Hilbert space projection.

$f\colon M \to \mathbb{C}$ $\qquad$ $L^2(M,\mathcal{S}) = L^2_+(M,\mathcal{S}) \oplus L^2_-(M,\mathcal{S})$

Let $D_f\colon L^2(M,\mathcal{S}) \to L^2(M,\mathcal{S})$ be $T_f \oplus I$.

i.e. $D_f\colon L^2(M,\mathcal{S}) \to L^2(M,\mathcal{S})$ is

$$D_f(s_1 + s_2) := T_f(s_1) + s_2 \qquad s_1 \in L^2_+(M,\mathcal{S}) \quad s_2 \in L^2_-(M,\mathcal{S})$$

D_f is a pseudo-differential operator of order zero.

Let n be any positive integer. Given a C^∞ map

$$\alpha\colon M \to GL(n, \mathbb{C})$$

view α as an $n \times n$ matrix $\alpha = [\alpha_{i,j}]$
where each $\alpha_{i,j}$ is a C^∞ function on M.

$$\alpha_{i,j}\colon M \to \mathbb{C}$$

Consider the operator $D_\alpha := [D_{\alpha_{i,j}}]$

$$D_\alpha\colon L^2(M, \mathcal{S})^{\oplus n} \to L^2(M, \mathcal{S})^{\oplus n}$$

D_α is an elliptic pseudo-differential operator of order zero.

The Atiyah-Singer formula for Index(D_α) is :

$$\text{Index}(D_\alpha) = (ch(\alpha) \cup Td(M))[M]$$

$$ch(\alpha) = \sum_{j \geq 0} \text{Tr}\left(\left(\frac{\alpha^{-1}d\alpha}{-2\pi i}\right)^{2j+1}\right)$$

Equivariant case of Atiyah-Singer

Families case of Atiyah-Singer

Proof of equivariant case and proof of families case :

Step 1. Dirac case via Bott periodicity.

Step 2. General case reduces to the Dirac case via a finite sequence of index-preserving moves.

Let G be a compact Lie group.

Definition

Define an abelian group, denoted $K_0^G(\cdot)$, by considering pairs (M, E) such that:

1. M is a closed even-dimensional C^∞ manifold with a given C^∞ action of G. $\quad G \times M \to M$.
2. A G-equivariant Spin^c structure for M is given.
3. E is a C^∞ G-equivariant $\mathbb{C}$ vector bundle on M.

"closed" = "compact without boundary"

Set $K_0^G(\cdot) = \{(M,E)\}/\sim$ where the the equivalence relation $\sim$ is generated by the three elementary steps :

bordism, direct sum-disjoint union, vector bundle modification.

Addition in $K_0^G(\cdot)$ is disjoint union.

$$K_0^G(\cdot) \text{ is an } R(G)\,\text{module.}$$

$$V \cdot (M,E) := (M, (M \times V) \otimes E)$$

<u>Notation.</u> $R(G)$ is the representation ring of the compact Lie group G.
V is a finite-dimensional representation of G. $\dim_{\mathbb{C}}(V) < \infty$.

Equivariant Bott periodicity implies that

$$\begin{aligned} K_0^G(\cdot) &\longrightarrow R(G) \\ (M, E) &\longmapsto \text{Index}(D_E) \end{aligned}$$

is an isomorphism of $R(G)$ modules.

For general equivariant case of Atiyah-Singer, consider

$$(S(T^*X \oplus 1_{\mathbb{R}}), E_\sigma) \in K_0^G(\cdot)$$

and use a finite sequence of index-preserving moves.

K-homology in algebraic geometry

Let X be a (possibly singular) projective algebraic variety / $\mathbb{C}$.

Grothendieck defined two abelian groups:

$K^0_{alg}(X) =$ Grothendieck group of algebraic vector bundles on X.

$K_0^{alg}(X) =$ Grothendieck group of coherent algebraic sheaves on X.

$K^0_{alg}(X) =$ the algebraic geometry K-theory of X (contravariant).

$K_0^{alg}(X) =$ the algebraic geometry K-homology of X (covariant).

K-homology in topology

Problem

How can K-homology be taken from algebraic geometry to topology?

K-homology is the dual theory to K-theory. There are three ways in which K-homology in topology has been defined:

Homotopy Theory K-theory is the cohomology theory and K-homology is the homology theory determined by the Bott (i.e. K-theory) spectrum.
This is the spectrum $\ldots, \mathbb{Z} \times BU, U, \mathbb{Z} \times BU, U, \ldots$

K-Cycles K-homology is the group of K-cycles.

C^*-algebras K-homology is the Atiyah-BDF-Kasparov group $KK^*(A, \mathbb{C})$.

Let X be a finite CW complex.
The three versions of K-homology are isomorphic.

$$K_j^{homotopy}(X) \underset{\longleftarrow}{\overset{\longrightarrow}{}} K_j(X) \longrightarrow KK^j(C(X), \mathbb{C})$$

homotopy theory K-cycles Atiyah-BDF-Kasparov

$$j = 0, 1$$

Cycles for K-homology

Let X be a CW complex.

Definition

A K-cycle on X is a triple (M, E, φ) such that :

1. M is a compact Spin^c manifold without boundary.
2. E is a $\mathbb{C}$ vector bundle on M.
3. $\varphi\colon M \to X$ is a continuous map from M to X.

Set $K_*(X) = \{(M, E, \varphi)\}/\sim$ where the equivalence relation $\sim$ is generated by the three elementary steps

- Bordism
- Direct sum - disjoint union
- Vector bundle modification

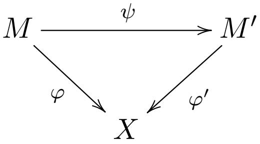
Isomorphism (M, E, φ) is isomorphic to (M', E', φ') iff $\exists$ a diffeomorphism

$$\psi\colon M \to M'$$

preserving the Spin^c-structures on M, M' and with

$$\psi^*(E') \cong E$$

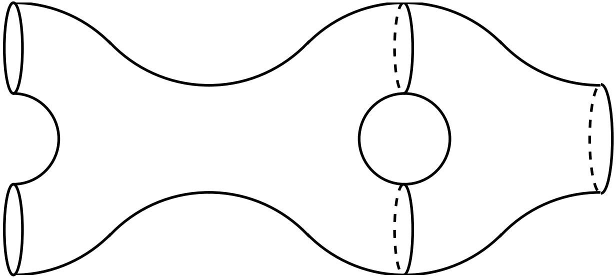
and commutativity in the diagram



Bordism (M_0, E_0, φ_0) is **bordant** to (M_1, E_1, φ_1) iff $\exists$ (Ω, E, φ) such that:

1. Ω is a compact Spin^c manifold with boundary.
2. E is $\mathbb{C}$ vector bundle on Ω.
3. $(\partial\Omega, E|_{\partial\Omega}, \varphi|_{\partial\Omega}) \cong (M_0, E_0, \varphi_0) \sqcup (-M_1, E_1, \varphi_1)$

$-M_1$ is M_1 with the Spin^c structure reversed.



(M_0, E_0, φ_0) $\qquad \downarrow \qquad$ $(-M_1, E_1, \varphi_1)$

X

Direct sum - disjoint union

Let E, E' be two $\mathbb{C}$ vector bundles on M

$$(M, E, \varphi) \sqcup (M, E', \varphi) \sim (M, E \oplus E', \varphi)$$

Vector bundle modification

(M, E, φ)

Let F be a Spin^c vector bundle on M

Assume that

$$\dim_{\mathbb{R}}(F_p) \equiv 0 \mod 2 \quad p \in M$$

for every fiber F_p of F

$$\mathbf{1}_{\mathbb{R}} = M \times \mathbb{R}$$

$$S(F \oplus \mathbf{1}_{\mathbb{R}}) := \text{unit sphere bundle of } F \oplus \mathbf{1}_{\mathbb{R}}$$

$$(M, E, \varphi) \sim (S(F \oplus \mathbf{1}_{\mathbb{R}}), \beta \otimes \pi^* E, \varphi \circ \pi)$$

$$\begin{array}{c} S(F \oplus \mathbf{1}_{\mathbb{R}}) \\ \Big\downarrow \pi \\ M \end{array}$$

This is a fibration with even-dimensional spheres as fibers.

$F \oplus \mathbf{1}_{\mathbb{R}}$ is a Spin^c vector bundle on M with odd-dimensional fibers. Let Σ be the spinor bundle for $F \oplus \mathbf{1}$

$$Cliff_{\mathbb{C}}(F_p \oplus \mathbb{R}) \otimes \Sigma_p \to \Sigma_p$$

$$\pi^*\Sigma = \beta \oplus \beta_-$$

$$(M, E, \varphi) \sim (S(F \oplus \mathbf{1}_{\mathbb{R}}), \beta \otimes \pi^* E, \varphi \circ \pi)$$

$$\{(M, E, \varphi)\}/\sim\; = K_0(X) \oplus K_1(X)$$

$K_j(X) =$ subgroup of $\{(M, E, \varphi)\}/\sim$ consisting of all (M, E, φ) such that every connected component of M has dimension $\equiv j \mod 2 \quad j = 0, 1$

Addition in $K_j(X)$ is disjoint union.

$$(M, E, \varphi) + (M', E', \varphi') = (M \sqcup M', E \sqcup E', \varphi \sqcup \varphi')$$

Additive inverse of (M, E, φ) is obtained by reversing the Spin^c structure of M.

$$-(M, E, \varphi) = (-M, E, \varphi)$$

Let X, Y be CW complexes and let $f\colon X \to Y$ be a continuous map.

Then $f_*\colon K_j(X) \to K_j(Y)$ is

$$f_*(M, E, \varphi) := (M, E, f \circ \varphi)$$

Atiyah-BDF-Kasparov K-homology

Reference. M.F. Atiyah, *Global Theory of Elliptic Operators*, Proc. Int. Conf. on Functional Analysis and Related Topics (Tokyo, 1969), University of Tokyo Press (1970).

M.F. Atiyah Brown-Douglas-Fillmore G.Kasparov

Let X be a finite CW complex.

$C(X) = \{\alpha : X \to \mathbb{C} \quad | \quad \alpha \text{ is continuous}\}$

$\mathcal{L}(\mathcal{H}) = \{\text{bounded operators } T : \mathcal{H} \to \mathcal{H}\}$

Any element in the Atiyah-BDF-Kasparov K-homology group $KK^0(C(X), \mathbb{C})$

is given by a 5-tuple $(\mathcal{H}_0, \psi_0, \mathcal{H}_1, \psi_1, T)$ such that :

- $\mathcal{H}_0$ and $\mathcal{H}_1$ are separable Hilbert spaces.
- $\psi_0 \colon C(X) \longrightarrow \mathcal{L}(\mathcal{H}_0)$ and $\psi_1 \colon C(X) \longrightarrow \mathcal{L}(\mathcal{H}_1)$ are unital $*$-homomorphisms.
- $T : \mathcal{H}_0 \longrightarrow \mathcal{H}_1$ is a (bounded) Fredholm operator.
- For every $\alpha \in C(X)$ the commutator $T \circ \psi_0(\alpha) - \psi_1(\alpha) \circ T$ $\in \mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$ is compact.

$$KK^0(C(X), \mathbb{C}) := \{(\mathcal{H}_0, \psi_0, \mathcal{H}_1, \psi_1, T)\}/ \sim$$

$$KK^0(C(X),\mathbb{C}) := \{(\mathcal{H}_0,\psi_0,\mathcal{H}_1,\psi_1,T)\}/\sim$$

$$(\mathcal{H}_0,\psi_0,\mathcal{H}_1,\psi_1,T)+(\mathcal{H}_0',\psi_0',\mathcal{H}_1',\psi_1',T') = (\mathcal{H}_0\oplus\mathcal{H}_0',\psi_0\oplus\psi_0',\mathcal{H}_1\oplus\mathcal{H}_1',\psi_1\oplus\psi_1',T\oplus T')$$

$$-(\mathcal{H}_0,\psi_0,\mathcal{H}_1,\psi_1,T) = (\mathcal{H}_1,\psi_1,\mathcal{H}_0,\psi_0,T^*)$$

Let X be a finite CW complex.
Any element in the Atiyah-BDF-Kasparov K-homology group $KK^1(C(X), \mathbb{C})$
is given by a 3-tuple $(\mathcal{H}, \psi, T)$ such that :

- $\mathcal{H}$ is a separable Hilbert space.
- $\psi\colon C(X) \longrightarrow \mathcal{L}(\mathcal{H})$ is a unital $*$-homomorphism.
- $T : \mathcal{H} \longrightarrow \mathcal{H}$ is a (bounded) self-adjoint Fredholm operator.
- For every $\alpha \in C(X)$ the commutator $T \circ \psi(\alpha) - \psi(\alpha) \circ T \in \mathcal{L}(\mathcal{H})$ is compact.

$$KK^1(C(X), \mathbb{C}) := \{(\mathcal{H}, \psi, T)\}/\sim$$

$$(\mathcal{H}, \psi, T) + (\mathcal{H}', \psi', T') = (\mathcal{H} \oplus \mathcal{H}', \psi \oplus \psi', T \oplus T')$$

$$-(\mathcal{H}, \psi, T) = (\mathcal{H}, \psi, -T)$$

Let X, Y be CW complexes and let $f\colon X \to Y$ be a continuous map.

Denote by $f^\natural\colon C(X) \leftarrow C(Y)$ the $*$-homomorphism

$$f^\natural(\alpha) := \alpha \circ f \qquad\qquad \alpha \in C(Y)$$

Then $f_*\colon KK^j(C(X), \mathbb{C}) \to KK^j(C(Y), \mathbb{C})$ is

$$f_*(\mathcal{H}, \psi, T) := (\mathcal{H}, \psi \circ f^\natural, T) \qquad\qquad j = 1$$

$$f_*(\mathcal{H}_0, \psi_0, \mathcal{H}_1, \psi_1, T) := (\mathcal{H}_0, \psi_0 \circ f^\natural, \mathcal{H}_1, \psi_1 \circ f^\natural, T) \qquad\qquad j = 0$$

Theorem (PB and R.Douglas and M.Taylor, PB and N. Higson and T. Schick)

Let X be a finite CW complex.

Then for $j = 0, 1$ the natural map of abelian groups

$$K_j(X) \to KK^j(C(X), \mathbb{C})$$

is an isomorphism.

For $j = 0, 1$ the natural map of abelian groups

$$K_j(X) \to KK^j(C(X), \mathbb{C})$$

is $(M, E, \varphi) \mapsto \varphi_*[D_E]$

where

1. D_E is the Dirac operator of M tensored with E.
2. $[D_E] \in KK^j(C(M), \mathbb{C})$ is the element in the Kasparov K-homology of M determined by D_E.
3. $\varphi_* \colon KK^j(C(M), \mathbb{C}) \to KK^j(C(X), \mathbb{C})$ is the homomorphism of abelian groups determined by $\varphi \colon M \to X$.

Let (M, E, φ) be a K-cycle on X, with M even-dimensional.

$$D_E \colon C^\infty(M, \mathcal{S}^+ \otimes E) \longrightarrow C^\infty(M, \mathcal{S}^- \otimes E)$$

Set $\mathcal{H}_0 = L^2(M, \mathcal{S}^+ \otimes E) \qquad \mathcal{H}_1 = L^2(M, \mathcal{S}^- \otimes E)$

For $j = 0, 1$ define $\psi_j \colon C(M) \to \mathcal{L}(\mathcal{H}_j)$ by :

$$\alpha \mapsto \mathcal{M}_\alpha \qquad \alpha \in C(M)$$

where $\mathcal{M}_\alpha$ is the multiplication operator

$$\mathcal{M}_\alpha(u) = \alpha u \qquad (\alpha u)(p) = \alpha(p)u(p) \qquad \alpha \in C(M), u \in \mathcal{H}_j, p \in M$$

Set $T = D_E(I + D_E^* D_E)^{-1/2}$ Then :

$$(\mathcal{H}_0, \psi_0, \mathcal{H}_1, \psi_1, T) \in KK^0(C(M), \mathbb{C})$$

and

$$\varphi_*(\mathcal{H}_0, \psi_0, \mathcal{H}_1, \psi_1, T) \in KK^0(C(X), \mathbb{C})$$

$$\varphi_*(\mathcal{H}_0, \psi_0, \mathcal{H}_1, \psi_1, T) := (\mathcal{H}_0, \psi_0 \circ \varphi^\natural, \mathcal{H}_1, \psi_1 \circ \varphi^\natural, T)$$

$$\varphi^\natural : C(M) \leftarrow C(X) \qquad \varphi^\natural(\gamma) := \gamma \circ \varphi \qquad \gamma \in C(X)$$

Let (M, E, φ) be a K-cycle on X, with M odd-dimensional.

$$D_E \colon C^\infty(M, \mathcal{S} \otimes E) \longrightarrow C^\infty(M, \mathcal{S} \otimes E)$$

Set $\mathcal{H} = L^2(M, \mathcal{S} \otimes E)$

Define $\psi \colon C(M) \to \mathcal{L}(\mathcal{H})$ by :

$$\alpha \mapsto \mathcal{M}_\alpha \qquad \alpha \in C(M)$$

where $\mathcal{M}_\alpha$ is the multiplication operator

$$\mathcal{M}_\alpha(u) = \alpha u \qquad (\alpha u)(p) = \alpha(p)u(p) \qquad \alpha \in C(M), u \in \mathcal{H}\,, p \in M$$

Set $T = D_E(I + D_E^* D_E)^{-1/2}$ Then :

$$(\mathcal{H}, \psi, T) \in KK^1(C(M), \mathbb{C})$$

and

$$\varphi_*(\mathcal{H}, \psi, T) \in KK^1(C(X), \mathbb{C})$$

$\varphi_*(\mathcal{H}, \psi, T) := (\mathcal{H}, \psi \circ \varphi^\natural, T)$

$\varphi^\natural \colon C(M) \leftarrow C(X)$ $\quad$ $\varphi^\natural(\gamma) := \gamma \circ \varphi$ $\quad$ $\gamma \in C(X)$

EXAMPLE. $S^1 \subset \mathbb{R}^2$

S^1 with its usual Spin^c structure has $\mathcal{S} = S^1 \times \mathbb{C}$.

The Dirac operator $D\colon L^2(S^1) \to L^2(S^1)$ is:

$$D = -i\frac{\partial}{\partial\theta}$$

The functions $e^{in\theta}$ are an orthonormal basis for $L^2(S^1)$.

Each $e^{in\theta}$ is an eigenvector of D :

$$-i\frac{\partial}{\partial\theta}(e^{in\theta}) = ne^{in\theta} \qquad n \in \mathbb{Z}$$

D is an unbounded self-adjoint operaror. $D^* = D$.

The bounded operator $T := D(I + D^*D)^{-1/2}$ is

$$T(e^{in\theta}) = \frac{n}{\sqrt{1+n^2}}e^{in\theta} \qquad n \in \mathbb{Z}$$

K-cycles and string theory

K-cycles are very closely connected to the D-branes of string theory. A D-brane is a K-cycle for the twisted K-homology of space-time.

In some models, the D-branes are allowed to evolve with time. This evolution is achieved by permitting the D-branes to change by the three elementary steps. Thus the underlying *charge* of a D-brane (i.e. the element in the twisted K-homology of space-time determined by the D-brane) remains unchanged as the D-brane evolves.

For more, see Jonathan Rosenberg's CBMS string theory lectures.
Also, see Baum-Carey-Wang paper *K-cycles for twisted K-homology* Journal of K-theory 12, 69-98, 2013.
Also, lectures (at Erwin Schrodinger Institute) by Bai-Ling Wang.

Comparison of $K_*(X)$ and $KK^*(C(X), \mathbb{C})$

Given some analytic data on X (i.e. an index problem) it is usually easy to construct an element in $KK^*(C(X), \mathbb{C})$. This does not solve the given index problem. $KK^*(C(X), \mathbb{C})$ does not have a simple explicitly defined chern character mapping it to $H_*(X; \mathbb{Q})$.

$K_*(X)$ does have a simple explicitly defined chern character mapping it to $H_*(X; \mathbb{Q})$.

$$ch\colon K_j(X) \longrightarrow \bigoplus_l H_{j+2l}(X; \mathbb{Q})$$

$$(M, E, \varphi) \mapsto \varphi_*(\mathrm{ch}(E) \cup Td(M) \cap [M])$$

With X a finite CW complex, suppose a datum (i.e. some analytical information) is given which then determines an element $\xi \in KK^j(C(X), \mathbb{C})$.

QUESTION : What does it mean to solve the index problem for ξ?

ANSWER : It means to explicitly construct the K-cycle (M, E, φ) such that

$$\mu(M, E, \varphi) = \xi$$

where $\mu \colon K_j(X) \to KK^j(C(X), \mathbb{C})$ is the natural map of abelian groups.

Suppose that $j = 0$ and that a K-cycle (M, E, φ) with

$$\mu(M, E, \varphi) = \xi$$

has been constructed. It then follows that for any $\mathbb{C}$ vector bundle F on X

$$\mathrm{Index}(F \otimes \xi) = \epsilon_*(ch(F) \cap ch(M, E, \varphi))$$

$\epsilon\colon X \longrightarrow \cdot$ $\qquad$ ϵ is the map of X to a point.

$$ch(M, E, \varphi) := \varphi_*(\mathrm{ch}(E) \cup Td(M) \cap [M])$$

REMARK. If the construction of the K-cycle (M, E, φ) with

$$\mu(M, E, \varphi) = \xi$$

has been done correctly, then it will work in the equivariant case and in the case of families of operators.

Example

General case of the Atiyah-Singer index theorem

Let X be a compact C^∞ manifold without boundary.
X is not required to be oriented.
X is not required to be even dimensional.
On X let

$$\delta : C^\infty(X, E_0) \longrightarrow C^\infty(X, E_1)$$

be an elliptic differential (or pseudo-differential) operator.
Then δ determines an element

$$[\delta] \in KK^0(C(X), \mathbb{C})$$

The K-cycle on X – which solves the index problem for δ – is:

$$(S(TX \oplus 1_{\mathbb{R}}), E_\sigma, \pi).$$

$$(S(TX \oplus 1_{\mathbb{R}}), E_\sigma, \pi)$$

$S(TX \oplus 1_{\mathbb{R}})$ is the unit sphere bundle of $TX \oplus 1_{\mathbb{R}}$.

$\pi\colon S(TX \oplus 1_{\mathbb{R}}) \longrightarrow X$ is the projection of $S(TX \oplus 1_{\mathbb{R}})$ onto X.

$S(TX \oplus 1_{\mathbb{R}})$ is even-dimensional and is a Spin^c manifold.

E_σ is the $\mathbb{C}$ vector bundle on $S(TX \oplus 1_{\mathbb{R}})$ obtained by doing a clutching construction using the symbol σ of δ.

$$\mu((S(TX \oplus 1_{\mathbb{R}}), E_\sigma, \pi)) = [\delta]$$

$$\Downarrow$$

$$\text{Index}(\delta) = (ch(E_\sigma) \cup Td(S(TX \oplus 1_{\mathbb{R}})))[(S(TX \oplus 1_{\mathbb{R}})]$$

which is the general Atiyah-Singer formula.

Next lecture : Tomorrow i.e. Thursday, 6 August.

Contact Manifolds

A contact manifold is an odd dimensional C^∞ manifold X
dimension$(X) = 2n + 1$
with a given C^∞ 1-form θ such that

$$\theta(d\theta)^n \text{ is non zero at every } x \in X - i.e.\ \theta(d\theta)^n \text{ is a volume form for } X.$$

Let X be a compact connected contact manifold without boundary $(\partial X = \emptyset)$.
Set dimension$(X) = 2n+1$.
Let r be a positive integer and let $\gamma\colon X \longrightarrow M(r, \mathbb{C})$ be a C^∞ map.
$M(r, \mathbb{C})$:= {r×r matrices of complex numbers}.

Assume: For each $x \in X$,
$\{\text{Eigenvalues of } \gamma(x)\} \cap \{\ldots, -n-4, -n-2, -n, n, n+2, n+4, \ldots\} = \emptyset$
i.e. $\forall x \in X$,
$\lambda \in \{\ldots -n-4, -n-2, -n, n, n+2, n+4, \ldots\} \Longrightarrow \lambda I_r - \gamma(x) \in GL(r, \mathbb{C})$

$\gamma\colon X \longrightarrow M(r,\mathbb{C})$
Are assuming : $\forall x \in X$,
$\lambda \in \{\ldots -n-4, -n-2, -n, n, n+2, n+4, \ldots\} \Longrightarrow \lambda I_r - \gamma(x) \in GL(r,\mathbb{C})$

Associated to γ is a differential operator P_γ which is hypoelliptic and Fredholm.

$$P_\gamma\colon C^\infty(X, X\times\mathbb{C}^r) \longrightarrow C^\infty(X, X\times\mathbb{C}^r)$$

P_γ is constructed as follows.

The sub-Laplacian Δ_H

Let H be the null-space of θ.

$$H = \{v \in TX \mid \theta(v) = 0\}$$

H is a C^∞ sub vector bundle of TX with

$$\text{For all } x \in X, \dim_{\mathbb{R}}(H_x) = 2n$$

The sub-Laplacian

$$\Delta_H \colon C^\infty(X) \to C^\infty(X)$$

is locally $-W_1^2 - W_2^2 - \cdots - W_{2n}^2$
where $W_1, W_2, \ldots, W_{2n}$ is a locally defined C^∞ orthonormal frame for H.
These locally defined operators are then patched together using a C^∞ partition of unity to give the sub-Laplacian Δ_H.

The Reeb vector field

The Reeb vector field is the unique C^∞ vector field W on X with :

$$\theta(W) = 1 \text{ and } \forall v \in TX,\ d\theta(W, v) = 0$$

Let

$$\gamma\colon X \longrightarrow M(r, \mathbb{C})$$

be as above, $P_\gamma\colon C^\infty(X, X \times \mathbb{C}^r) \to C^\infty(X, X \times \mathbb{C}^r)$ is defined:

$$P_\gamma = i\gamma(W \otimes I_r) + (\Delta_H) \otimes I_r \qquad I_r = r \times r \text{ identity matrix} \quad i = \sqrt{-1}$$

P_γ is a differential operator (of order 2) and is hypoelliptic but not elliptic.

These operators P_γ have been studied by :

- R.Beals and P.Greiner *Calculus on Heisenberg Manifolds* Annals of Math. Studies 119 (1988).
- C.Epstein and R.Melrose.
- E. van Erp *The Atiyah-Singer index formula for subelliptic operators on contact manifolds. Part 1 and Part 2* Annals of Math. 171(2010).

A class of operators with somewhat similar analytic and topological properties has been studied by A. Connes and H. Moscovici.
M. Hilsum and G. Skandalis.

Set $T_\gamma = P_\gamma(I + P_\gamma^* P_\gamma)^{-1/2}$.

Let $\psi\colon C(X) \to \mathcal{L}(L^2(X) \otimes_{\mathbb{C}} \mathbb{C}^r)$ be

$$\psi(\alpha)(u_1, u_2, \ldots, u_r) = (\alpha u_1, \alpha u_2, \ldots, \alpha u_r)$$

where for $x \in X$ and $u \in L^2(X), (\alpha u)(x) = \alpha(x)u(x)$

$$\alpha \in C(X) \qquad\qquad u \in L^2(X)$$

Then

$$(L^2(X) \otimes_{\mathbb{C}} \mathbb{C}^r,\ \psi,\ \ L^2(X) \otimes_{\mathbb{C}} \mathbb{C}^r,\ \psi,\ \ T_\gamma) \in KK^0(C(X), \mathbb{C})$$

Denote this element of $KK^0(C(X), \mathbb{C})$ by $[P_\gamma]$.

$$[P_\gamma] \in KK^0(C(X), \mathbb{C})$$

$$[P_\gamma] \in KK^0(C(X), \mathbb{C})$$

QUESTION.What is the K-cycle that solves the index problem for $[P_\gamma]$?

ANSWER. To construct this K-cycle, first recall that the given 1-form θ which makes X a contact manifold also makes X a stably almost complex manifold :

$$(\text{contact}) \Longrightarrow (\text{stably almost complex})$$

(contact) $\Longrightarrow$ (stably almost complex)

Let θ, H, and W be as above. Then :

- $TX = H \oplus 1_{\mathbb{R}}$ where $1_{\mathbb{R}}$ is the (trivial) $\mathbb{R}$ line bundle spanned by W.
- A morphism of C^{∞} $\mathbb{R}$ vector bundles $J : H \to H$ can be chosen with $J^2 = -I$ and $\forall x \in X$ and $u, v \in H_x$

$$d\theta(Ju, Jv) = d\theta(u, v) \qquad d\theta(Ju, u) \geq 0$$

- J is unique up to homotopy.

(contact) $\Longrightarrow$ (stably almost complex)

$J\colon H \to H$ is unique up to homotopy.
Once J has been chosen :

H is a C^∞ $\mathbb{C}$ vector bundle on X.

$$\Downarrow$$

$TX \oplus 1_{\mathbb{R}} = H \oplus 1_{\mathbb{R}} \oplus 1_{\mathbb{R}} = H \oplus 1_{\mathbb{C}}$ is a C^∞ $\mathbb{C}$ vector bundle on X.

$$\Downarrow$$

$$X \times S^1 \text{ is an almost complex manifold.}$$

REMARK. An almost complex manifold is a $\mathbb{C}^\infty$ manifold Ω with a given morphism $\zeta\colon T\Omega \to T\Omega$ of C^∞ $\mathbb{R}$ vector bundles on Ω such that

$$\zeta \circ \zeta = -I$$

The conjugate almost complex manifold is Ω with ζ replaced by $-\zeta$.

NOTATION. As above $X \times S^1$ is an almost complex manifold, $\overline{X \times S^1}$ denotes the conjugate almost complex manifold.

Since (almost complex)$\Longrightarrow$ (Spin^c), the disjoint union $X \times S^1 \sqcup \overline{X \times S^1}$ can be viewed as a Spin^c manifold.

Let

$$\pi \colon X \times S^1 \sqcup \overline{X \times S^1} \longrightarrow X$$

be the evident projection of $X \times S^1 \sqcup \overline{X \times S^1}$ onto X.
i.e.

$$\pi(x, \lambda) = x \qquad\qquad (x, \lambda) \in X \times S^1 \sqcup \overline{X \times S^1}$$

The solution K-cycle for $[P_\gamma]$ is $(X \times S^1 \sqcup \overline{X \times S^1},\ E_\gamma,\ \pi)$

$$E_\gamma = \Big(\bigoplus_{j=0}^{j=N} L(\gamma, n{+}2j)\otimes\pi^*\mathrm{Sym}^j(H)\Big)\bigsqcup\Big(\bigoplus_{j=0}^{j=N} L(\gamma, -n{-}2j)\otimes\pi^*\mathrm{Sym}^j(H^*)\Big)$$

1. "Sym^j" is " j-th symmetric power".
2. H^* is the dual vector bundle of H.
3. N is any positive integer such that : $n + 2N > \sup\{||\gamma(x)||, x \in X\}$.
4. $L(\gamma, n + 2j)$ is the $\mathbb{C}$ vector bundle on $X \times S^1$ obtained by doing a clutching construction using $(n + 2j)I_r - \gamma\colon X \to GL(r, \mathbb{C})$.
5. Similarly, $L(\gamma, -n - 2j)$ is obtained by doing a clutching construction using $(-n - 2j)I_r - \gamma\colon X \to GL(r, \mathbb{C})$.

Restriction of E_γ to $X \times S^1$

Let N be any positive integer such that :

$$n + 2N > \sup\{||\gamma(x)||, x \in X\}$$

The restriction of E_γ to $X \times S^1$ is:

$$E_\gamma \mid X \times S^1 = \bigoplus_{j=0}^{j=N} L(\gamma, n + 2j) \otimes \pi^* \mathrm{Sym}^j(H)$$

Restriction of E_γ to $\overline{X \times S^1}$

The restriction of E_γ to $\overline{X \times S^1}$ is:

$$E_\gamma \mid \overline{X \times S^1} = \bigoplus_{j=0}^{j=N} L(\gamma, -n-2j) \otimes \pi^* \mathrm{Sym}^j(H^*)$$

Here H^* is the dual vector bundle of H:

$$H_x^* = \mathrm{Hom}_{\mathbb{C}}(H_x, \mathbb{C}) \qquad\qquad x \in X$$

$$E_\gamma = \Big(\bigoplus_{j=0}^{j=N} L(\gamma, n{+}2j){\otimes}\pi^*\mathrm{Sym}^j(H)\Big) \bigsqcup \Big(\bigoplus_{j=0}^{j=N} L(\gamma, -n{-}2j){\otimes}\pi^*\mathrm{Sym}^j(H^*)\Big)$$

Theorem (PB and Erik van Erp)

$$\mu(X \times S^1 \sqcup \overline{X \times S^1},\ E_\gamma\,,\ \pi) = [P_\gamma]$$